Tutorial Notes Information

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- Prerequisite Knowledge:
  - Differentiation of Vector Valued Functions
  - Inner Product & Cross Product
  - Differentiation By Part
- Table of Content & Outline
  - Formulae Regarding $\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau$
  - Key Techniques for Proofs involving Frenet Serret Equations
  - MATH 4030 Differential Geometry Pastpaper
- References:
  - **Lecture Notes** of Dr. LAU & Dr. CHENG
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 Introduction

This document serves as a summary sheet & revision checklist for you, since Theory of Curves and Frenet Serret Equation involve quite a lot of formulae and special techniques. Instead of pure lecturing during tutorial, getting your hands dirty probably enhance your learning efficiency, while cater for learning diversities.

2 Notations

1. $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a space curve (not necessarily parameterised by arc-length)
   - $\mathbf{r}(t)$ denotes a space curve **not necessarily** parametrized by arc-length
   - $\mathbf{r}(s)$ denotes a space curve parametrized by arc-length ($\|\mathbf{r}'(s)\| = 1$)
2. $\mathbf{T}$ denotes the unit tangent of $\mathbf{r}$
3. $\mathbf{N}$ denotes the unit normal of $\mathbf{r}$
4. $\mathbf{B}$ denotes the unit binormal of $\mathbf{r}$
5. κ denotes the curvature of $\mathbf{r}$
6. τ denotes the torsion of $\mathbf{r}$

3 Formulae of T, N, B Frame

(1). $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ and $\mathbf{T}(s) = \mathbf{r}'(s)$ $\quad$ $[\mathbf{T}'(s) = \mathbf{r}''(s)]$

(2). $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ and $\mathbf{N}(s) = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \dfrac{\mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$

(3). $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ and $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$

4 Formulae of Curvature κ

(1). $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

(2). $\kappa(s) = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$

(3). $\kappa(t) = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

(4). $\kappa(t) = \dfrac{|x'(t)y''(t) - x''(t)y'(t)|}{|x'(t)^2 + y'(t)^2|^{3/2}}$, for plane curve $\mathbf{r}(t) = (x(t), y(t))$

(5). $\kappa(x) = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$, for graph of function $y = f(x)$

(6). $\kappa(\theta) = \dfrac{|r(\theta)^2 + 2r'(\theta)^2 - r(\theta)r''(\theta)|}{(r(\theta)^2 + r'(\theta)^2)^{3/2}}$, for $\mathbb{R}^2$ polar graph $\mathbf{r}(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$

(7). Signed Curvature for plane curve: $\begin{cases} \theta(s) = \arctan\left(\dfrac{y'(t)}{x'(t)}\right) = \text{Angle between } \mathbf{T} \text{ and positive } x\text{-axis} \\ \\ \kappa(t) = \dfrac{d\theta}{dt} = \dfrac{x'(t)y''(t) - x''(t)y'(t)}{|x'(t)^2 + y'(t)^2|^{3/2}} \end{cases}$

Note: When $\kappa(t) = 0$, then θ reaches local extremum (point of inflexion of $\mathbf{r}$?)

5 Formulae for Torsion τ

(1). $\tau(t) = \left\langle \dfrac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$ and $\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle = -\langle \mathbf{N}(s), \mathbf{B}'(s) \rangle$

(2). $\tau(t) = \dfrac{\langle \mathbf{r}'(t) \times \mathbf{r}''(t), \mathbf{r}'''(t) \rangle}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$

6 Facts about Curvature κ & Torsion τ

(1). $\kappa = 0 \iff$ Straight Line (Prop. 2.3.4 of Lecture Notes)

Note: Torsion of a zero-curvature curve is undefined

(2). Constant κ and $\tau = 0 \Rightarrow$ Circle with radius $\dfrac{1}{\kappa}$

(3). $\kappa > 0$ and $\tau = 0 \iff$ Plane Curve (contained in a plane) (Prop 2.4.6 of Lecture Notes)

7 Frenet Serret Equation

(1). $\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$

(2). $\begin{pmatrix} \mathbf{T}'(t) \\ \mathbf{N}'(t) \\ \mathbf{B}'(t) \end{pmatrix} = \begin{pmatrix} 0 & \|\mathbf{r}'(t)\|\kappa(t) & 0 \\ -\|\mathbf{r}'(t)\|\kappa(t) & 0 & \|\mathbf{r}'(t)\|\tau(t) \\ 0 & -\|\mathbf{r}'(t)\|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{pmatrix}$

8 Key Techniques for Proofs involving Frenet Serret Equations

8.1 T, N, B Constitutes Orthonormal Basis

Example 8.1 (Problem Set Exercise 3.4 (Lecture Notes Chapter 2 Exercise 18)).

Given that $\langle \mathbf{u}, \mathbf{N}(s)\rangle = 0$, let $\mathbf{u} \in \mathbb{R}^3$, then:

$$\begin{aligned}\mathbf{u} &= \langle \mathbf{u}, \mathbf{T}(s)\rangle\, \mathbf{T}(s) + \langle \mathbf{u}, \mathbf{N}(s)\rangle\, \mathbf{N}(s) + \langle \mathbf{u}, \mathbf{B}(s)\rangle\, \mathbf{B}(s) \\ &= \langle \mathbf{u}, \mathbf{T}(s)\rangle\, \mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s)\rangle\, \mathbf{B}(s)\end{aligned}$$

Strategy: Orthonormal Basis & Taking Inner Product at Both Sides

8.2 T, N, B are Linearly Independent

Example 8.2 (Problem Set Exercise 4.1 (2014 TDG Final Exam) (2019 TDG Quiz 2)).

Given that $(\lambda'(s) - 1)\, \mathbf{T}(s) + [\lambda(s)\kappa(s) - \mu(s)\tau(s)]\, \mathbf{N}(s) + \mu'(s)\, \mathbf{B}(s) = 0$

Since $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$ is a orthnormal basis for $\mathbb{R}^3$, they are linearly independent

Hence we have: $\begin{cases} \lambda'(s) - 1 = 0 \\ \lambda(s)\kappa(s) - \mu(s)\tau(s) = 0 \\ \mu'(s) = 0 \end{cases}$

Strategy: If $a(t)\, \mathbf{T}(t) + b(t)\, \mathbf{N}(t) + c(t)\, \mathbf{B}(t) = \mathbf{0}$, then $a(t) = b(t) = c(t) \equiv 0$

8.3 Computation of Norm with T, N, B

Example 8.3 (Problem Set Exercise 3.4 (Lecture Notes Chapter 2 Exercise 18)).

Given that $\mathbf{u} = \langle\langle \mathbf{u}, \mathbf{T}(s)\rangle\, \mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s)\rangle\, \mathbf{B}(s)$

$$\begin{aligned}\|\mathbf{u}\|^2 &= \Big\langle \langle \mathbf{u}, \mathbf{T}(s)\rangle\, \mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s)\rangle\, \mathbf{B}(s), \langle \mathbf{u}, \mathbf{T}(s)\rangle\, \mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s)\rangle\, \mathbf{B}(s) \Big\rangle \\ &= \langle \mathbf{u}, \mathbf{T}(s)\rangle^2 \langle \mathbf{T}(s)\, \mathbf{T}(s)\rangle + 0 + \langle \mathbf{u}, \mathbf{B}(s)\rangle^2 \langle \mathbf{B}(s), \mathbf{B}(s)\rangle \\ &= \langle \mathbf{u}, \mathbf{T}(s)\rangle^2 + \langle \mathbf{u}, \mathbf{B}(s)\rangle^2\end{aligned}$$

Strategy: If $\mathbf{u}(t) = a(t)\, \mathbf{T}(t) + b(t)\, \mathbf{N}(t) + c(t)\, \mathbf{B}(t)$, then $\|\mathbf{u}(t)\| = a(t)^2 + b(t)^2 + c(t)^2$
Moreover, by orthonormal basis, $a(t) = \langle \mathbf{u}(t), \mathbf{T}(t)\rangle$, $b(t) = \langle \mathbf{u}(t), \mathbf{N}(t)\rangle$, $c(t) = \langle \mathbf{u}(t), \mathbf{B}(t)\rangle$

8.4 T / N / B Always Passes Through Point

Example 8.4 (DG 2017 MT).

Given that all tangent lines of $\mathbf{r}$ pass through a fixed point $p_0 \in \mathbb{R}^2$

$$\exists l : \mathbb{R} \to \mathbb{R} \text{ such that } \forall s \in \mathbb{R}, \mathbf{r}(s) + l(s)\, \mathbf{T}(s) = p_0$$

Strategy: $\exists l(s)$ such that $\forall s \in \mathbb{R}, \mathbf{r}(s) + f(s)\mathbf{V}(s) = p_0$, where $\mathbf{V} = \mathbf{T}, \mathbf{N}, \mathbf{B}$

9 Computational Questions from Frenet Frame Problem Set

Exercise 9.1 ((a) 2018 TDG Quiz 2 — (b) 2020 TDG Quiz 2).
Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ of the space curves below.

(a). $\alpha(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta)\ ,\ \theta \in \mathbb{R}$

(b). $\alpha(\theta) = \left(t - \sin t\cos t,\ \sin^2 t,\ \cos t\right)\ ,\ t \in (0, \pi)$

10 You can score 100 in University DG Assessments !

Exercise 10.1 (DG 2017 MT).
Let $\alpha : (-1.1) \to \mathbb{R}^3$ be the space curve given by:

$$\alpha(s) = \left(\frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{1}{\sqrt{2}}s\right)$$

Compute $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and κ and τ

Exercise 10.2 (DG 2017 MT).
Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a plane curve parametrized by arc-length, such that all tangent lines of $\mathbf{r}$ pass through a fixed point $p_0 \in \mathbb{R}^2$. Show that α must be a straight line passing through the point p_0

Exercise 10.3 (2016 DG MT).
Let $\mathbf{r} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a space curve parametrized by arc-length with curvature $\kappa(s) > 0$ for all $s \in I$. Suppose that the trace of $\mathbf{r}$ is contained in the unit sphere $\mathbb{S}^2$ and that $\mathbf{r}$ has constant torsion $\tau(s) \equiv a$. By differentiating by part twice $\|\mathbf{r}(s)\|^2 = 1$, prove that there exists constants $b, c, \in \mathbb{R}$ such that

$$\kappa(s) = \frac{1}{b\cos(as) + c\sin(as)}$$

Exercise 10.4. Still not satisfied with the problems ? Try out Frenet Frame Problem Set∼

Exercise 10.5. Reflect on your TDG Quiz 1 (non-academic) Performance:

1. Time Management
2. Strategy of Choosing Question to Answer
3. Brute Force or Clever Method ?
4. Skip Steps // Too Many Unnecessary Steps ?
5. Struggling on a single question for 10 minutes ?